ON r-FOLD SYMMETRY OF PLANE ALGEBRAIC CURVES.*

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If a plane curve is revolved about a point in its own plane through an angle of $360°/r$ and if it then coincides with its former position, it is said to have r-fold symmetry with respect to the point; and the point is called the center of r-fold symmetry. The object of this paper is to ascertain the analytical conditions which are necessary and sufficient to the existence of r-fold symmetry and to examine into the geometric properties of the curves in certain special cases. In a previous note† I have given a classification of plane algebraic curves having four-fold symmetry about a point, and this has been followed‡ by a paper on the geometric properties of quartic curves possessing such four-fold symmetry.

In the present discussion we shall confine ourselves to plain algebraic loci which are such that no locus is composed entirely of isolated points or of straight lines; in other words, every locus considered will be assumed to have at least one part which is *continuous* and *curved.* And this assumption is made throughout without further statement.

Evidently the circle is a curve of infinite-fold symmetry. It is clear that the condition of infinite-fold symmetry with respect to the origin is that the polar equation shall be independent of the vectorial angle; that is, the locus in this case is a circle or a system of concentric circles with center at the center of infinite-fold symmetry. Therefore it will be sufficient in what follows to confine our attention to the cases in which r is finite.

1. Separation into two classes. Let n be the order of a curve of r-fold symmetry and let it be referred to rectangular cartesian coordinates with origin at the center of r-fold symmetry. Take the equation in the form

$$\Sigma\, a_{ts}x^t y^s = 0, \tag{1}$$

where a_{ts} is a real constant for every t and s and where t and s each range over the values 0, 1, 2, ..., n subject to the condition

$$t+s \leqq n.$$

Certain relations must exist among the coefficients a. These are now to be found.

If we transform equation (1) by the substitution

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‡*Annals of Mathematics*, Vol. 10, No. 2, pp. 81-87.

$$x=\rho \cos \theta, \quad y=\rho \sin \theta,$$

we have

(2) $$\sum_{v=0}^{v=n} \rho^v \sum_{t,\, s=0}^{t,\, s=v} a_{ts} \cos^t \theta \sin^s \theta=0, \quad (t+s=\nu).$$

Putting $\theta=\theta_1$ and $\theta=\theta_1+\alpha\,\phi$ where

$$\phi=360^\circ/r \text{ and } \alpha=\text{an integer},$$

we have the following equations:

(3) $$\sum_{v=0}^{v=n} \rho^v \sum_{t,\, s=0}^{t,\, s=v} a_{ts} \cos^t \theta_1 \sin^s \theta_1=0, \quad (t+s=\nu),$$

(4) $$\sum_{v=0}^{v=n} \rho^v \sum_{t,\, s=0}^{t,\, s=v} a_{ts} \cos^t (\theta_1+\alpha\,\phi) \sin^s (\theta+\alpha\,\phi)=0, \quad (t+s=\nu).$$

From the existence of the defined r-fold symmetry it follows that equations (3) and (4) must yield by solution the same values of ρ. Therefore the coefficients can differ only by a constant factor m_α; that is,

(5) $$\sum_{t,\, s=0}^{t;\, s=v} a_{ts} \cos^t \theta_1 \sin^s \theta_1=m_\alpha \sum_{t,\, s-0}^{t,\, s=v} a_{ts} \cos^t (\theta_1+\alpha\,\phi)\sin^s (\theta_1+\alpha\,\phi), \quad (t+s=\nu).$$

Equation (5) must hold for each value of ν from 0 to n and for each value of α from 1 to r, a different equation being formed for every case. Then (5) yields $r(n+1)$ equations which must all be satisfied for every possible value of θ_1. It is clear that the existence of this system of equations is both necessary and sufficient to the existence of the defined symmetry.

We shall now evaluate the constants m_α. If we take $\alpha=1$ and multiply equation (4) by m_1 (which evidently cannot be zero) it follows that the result is identical with equation (3). Hence a second addition of ϕ to the vectorial angle would necessitate a second multiplication of the coefficients by m_1; that is, two multiplications by m_1 produces m_2; or

$$m_2=m_1^2.$$

Continuing the additions of ϕ to the vectorial angle, we have

$$m_2=m_1^2, \quad m_3=m_1^3, \quad \ldots, \quad m_r=m_1^r.$$

At the rth addition of ϕ to the vectorial angle the two members of (5) become identical except as to the presence of the factor m_r in the second member; and therefore m_r must equal 1. For, if not, we must have

$$\Sigma\, a_{ts} \cos^t \theta_1 \sin^s \theta_1 = 0, \quad (t+s=\nu),$$

for an unlimited number of values of θ_1 each less than 2π; and this is evidently impossible. Since $m_r = m_1{}^r$ and $m_r = 1$, we have $m_1{}^r = 1$. It is evident from (5) that m_1 is a real quantity. Therefore

$$m_1 = +1, \text{ when } r \text{ is odd}; \tag{6}$$
$$m_1 = \pm 1, \text{ when } r \text{ is even}; \tag{7}$$
$$m_a = m_1{}^a, \text{ in every case}. \tag{8}$$

In order to find the necessary and sufficient relations among the coefficients a we proceed as follows. For $m_1 = +1$, $m_1 = -1$, equation (5) takes the respective forms:

$$\text{(9)} \quad \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t\theta_1 \sin^s\theta_1 = \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t(\theta_1 + a\,\phi) \sin^s(\theta_1 + a\,\phi), \ (t+s=\nu), \ [A],$$

$$\text{(10)} \quad \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t\theta_1 \sin^s\theta_1 = (-1)^a \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t(\theta_1 + a\,\phi) \sin^s(\theta_1 + a\,\phi), \ (t+s=\nu), \ [B],$$

where ν ranges over all the values 0, 1, 2, ..., n, different equations being formed for each value of ν. Equation (9) alone holds when r is odd; when r is even both equations (9) and (10) may hold. Evidently these equations are necessary and sufficient to the existence of r-fold symmetry; that is, for odd-fold symmetry we must be able to satisfy (9); for even-fold symmetry, either (9) or (10) or both. In case this condition cannot be satisfied for given r and n, we are to conclude that r-fold symmetry does not exist for curves of such degree n. As a case in point, we have the theorem: Four-fold symmetry does not exist for curves of odd degree.*

We shall say that curves which satisfy equations (9) and (10) are of class A and B, respectively. In class A there will be found loci of both odd- and even-fold symmetry; in class B will be found loci of only even-fold symmetry.

Obviously, if the equation of a curve referred to rectangular axes has only terms of even degree or only terms of odd degree, the curve has two-fold symmetry; for in either case, if α, β is a point on the curve, so is $-\alpha$,

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$-\beta$. Conversely, if the origin of rectangular coordinates is taken at the center of two-fold symmetry, the equation must evidently have one of the forms indicated. If each term is of even degree, it is obvious that the curve belongs to class A; while if each term is of odd degree, the curve belongs to class B.

2. Determination of constants a_{ts} for class A. Equation (9) indicates that the real function

$$F \equiv \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1, \quad (t+s=\nu), \tag{11}$$

is periodic with the real period $\phi=2\pi/r$. But every real function of a single variable θ_1 with the real period $2\pi/r$ can be expanded in a Fourier series in the general form

$$\sum_{i=0}^{\infty} c_i \cos ir\,\theta_1 + \sum_{i=1}^{\infty} y_i \sin ir\,\theta_1. \tag{12}$$

If $\cos ir\,\theta_1$ and $\sin ir\,\theta_1$ are expanded in terms of $\sin\theta_1$ and $\cos\theta_1$ the results are homogeneous of order ir in $\sin\theta_1$, $\cos\theta_1$; moreover, the coefficients c_i and y_i do not belong to terms alike in $\sin\theta_1$ and $\cos\theta_1$, and therefore cannot annul each other. Hence, if the expression in (12) is to be identical with F, $ir \leqq \nu$. If $ir=\nu-2j$, the corresponding part of (12) when expanded in terms of $\cos\theta_1$, $\sin\theta_1$ is of degree $\nu-2j$; but it becomes of degree ν through multiplication by the unit factor $(\cos^2\theta_1+\sin^2\theta_1)^j$. Evidently ir cannot differ from ν by an odd number. Hence, as a result we have

$$\sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = \sum_{i=0}^{i} c_i \cos ir\,\theta_1 + \sum_{i=1}^{i} y_i \sin ir\,\theta_1, \tag{13}$$

where $t+s=\nu$ and ir is always positive and has as its values some or all of the positive numbers of the series ν, $\nu-2$, $\nu-4$, ...

Now, if ν ranges from 0 to n, the preceding result enables us to determine readily the values of all of the coefficients a_{ts} in terms of a suitable number of them selected as independent constants. Substituting these values in (1) we obtain the most general form of the equation of the nth degree locus possessing r-fold symmetry and belonging to class A as defined above. Such equations, for several values of r and n, are written out below in their most general form.*

*For curves of four-fold symmetry see my previous papers already referred to. Curves of two-fold symmetry are disposed of at the close of section 1 of this paper.

SOME CURVES OF CLASS A OF 3-FOLD SYMMETRY.

$F_2 \equiv c_1 + c_2 (x^2 + y^2) = 0.$

$F_3 \equiv F_2 + c_3 x^3 + c_4 y^3 - 3c_4 x^2 y - 3c_3 x y^2 = 0.$

$F_4 \equiv F_3 + c_5 (x^2 + y^2)^2 = 0.$

$F_5 \equiv F_4 + c_6 x^5 + c_7 y^5 - 3c_7 x^4 y - 3c_6 x y^4 - 2c_6 x^3 y^2 - 2c_7 x^2 y^3 = 0.$

$$F_6 \equiv F_5 + c_8 (x^6 - y^6) + 6c_9 xy(x^4 + y^4) - 15c_8 x^2 y^2 (x^2 - y^2) - 20c_9 x^3 y^3 + c_{10} (x^2 + y^2)^3 = 0.$$

$F_7 \equiv F_6 + (x^2 + y^2)(c_{11} x^5 + c_{12} y^5 - 3c_{12} x^4 y - 2c_{11} x^3 y^2 - 2c_{12} x^2 y^3) = 0.$

SOME CURVES OF CLASS A OF 5-FOLD SYMMETRY.

$F_2 \equiv c_1 + c_2 (x^2 + y^2) = 0.$

$F_4 \equiv F_2 + c_3 (x^2 + y^2)^2 = 0.$

$F_5 \equiv F_4 + c_4 x^5 + c_5 y^5 + 5c_5 x^4 y + 5c_4 x y^4 - 10c_4 x^3 y^2 - 10c_5 x^2 y^3 = 0.$

$F_6 \equiv F_5 + c_6 (x^2 + y^2)^3 = 0.$

$$F_7 \equiv F_6 + (x^2 + y^2)(c_7 x^5 + c_8 y^5 + 5c_8 x^4 y + 5c_7 x y^4 - 10c_7 x^3 y^2 - 10c_8 x^2 y^3 = 0.$$

SOME CURVES OF CLASS A OF 6-FOLD SYMMETRY.

$F_2 \equiv c_1 + c_2 (x^2 + y^2) = 0.$

$F_4 \equiv F_2 + c_3 (x^2 + y^2)^2 = 0.$

$$F_6 \equiv F_4 + c_4 (x^2 + y^2)^3 + c_5 (x^6 - y^6) + 6c_6 xy(x^4 + y^4) - 15c_5 x^2 y^2 (x^2 - y^2) - 20c_6 x^3 y^3 = 0.$$

SOME CURVES OF CLASS A OF 7-FOLD SYMMETRY.

$F_2 \equiv c_1 + c_2 (x^2 + y^2) = 0.$

$F_4 \equiv F_2 + c_3 (x^2 + y^2)^2 = 0.$

$F_6 \equiv F_4 + c_4 (x^2 + y^2)^3 = 0.$

$$F_7 \equiv F_6 + c_5 x^7 + c_6 y^7 - 7c_6 x^6 y - 7c_5 x y^6 - 21c_5 x^5 y^2 - 21c_6 x^2 y^5 + 35c_6 x^4 y^3 + 35c_5 x^3 y^4 = 0.$$

3. Determination of the constants a_{ts} for class B. For class B we have seen that r is even. It may be shown that r-fold symmetry in class B is a special case of $\frac{1}{2}r$-fold symmetry in class A. For if ϕ is the angle through which the r-fold (r even) symmetrical curve of class B must be turned in order to coincide with its original position, 2ϕ is the angle through which the $\frac{1}{2}r$-fold symmetrical curve of class A must be turned that it may coincide with its original position. But the former still coincides with its first posi-

tion after being turned through an angle of 2ϕ. Hence, since r must be even for curves of class B, r-fold symmetry of class B is a special case of $\frac{1}{2}r$-fold symmetry of class A.

For $\frac{1}{2}r$-fold symmetry equation (13) becomes

$$(14) \qquad \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = \sum_{i=0}^{i} c_i \cos\frac{ir}{2}\theta_1 + \sum_{i=1}^{i} y_i \sin\frac{ir}{2}\theta_1,$$

where $t+s=\nu$ and $ir/2$ is always positive and has as its values some or all the positive numbers of the series ν, $\nu-2$, $\nu-4$, ... This is a necessary condition for r-fold symmetry in class B. From (10) we may write

$$(15) \qquad \sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = -\sum_{t,\,s=0}^{t,\,s=\nu} a_{ts} \cos^t (\theta_1+\phi) \sin^s (\theta_1+\phi), \quad (t+s=\nu).$$

Since the existence of equation (10) is a necessary and sufficient condition for r-fold symmetry of curves of class B, it is readily seen from the discussion in the preceding paragraph that the existence at the same time of equations (14) and (15) is also a necessary and sufficient condition for r-fold symmetry of curves of class B. This result enables one to determine the constants a_{ts} in terms of a suitable number of them chosen as independent constants.

But if r is twice an odd number, the constants may be more readily determined in the following manner: In the equation for the curve of class A of $\frac{1}{2}r$-fold symmetry, insert the condition for two-fold symmetry of class B; that is, let the equation consist only of terms of odd degree. In this way were found the equations for 6- and 10-fold symmetry given below.

Some Curves of Class B of 6-fold Symmetry.

$$F_3 \equiv c_1x^3+c_2y^3-3c_2x^2y-3c_1xy^2=0.$$

$$F_5 \equiv F_3+c_3x^5+c_4y^5-3c_4x^4y-3c_3xy^4-2c_3x^3y^2-2c_4x^2y^3=0.$$

$$F_7 \equiv F_5+(x^2+y^2)(c_5x^5+c_6y^5-3c_6x^4y-3c_5xy^4-2c_5x^3y^2-2c_6x^2y^3) \\ +(x^2+y^2)^2(c_7x^3+c_8y^3-3c_8x^2y-3c_7xy^2)=0.$$

Some Curves of Class B of 10-fold Symmetry.

$$F_5 \equiv c_1x^5+c_2y^5+5c_2x^4y+5c_1xy^4-10c_1x^3y^2-10c_2x^2y^3=0.$$

$$F_7 \equiv F_5+(x^2+y^2)(c_3x^5+c_4y^5+5c_4x^4y+5c_3xy^4-10c_3x^3y^2 \\ -10c_4x^2y^3)=0.$$

$$F_9 \equiv F_7+(x^2+y^2)^2(c_5x^5+c_6y^5+5c_6x^4y+5c_5xy^4-10c_5x^3y^2 \\ -10c_6x^2y^3)=0.$$

A different method, however, is necessary for curves of 8-fold symmetry. In this case we must employ equations (14) and (15); or, what is the same thing, a corollary from my first paper on four-fold symmetry (already referred to) and equation (15): namely, the necessary and sufficient condition for four-fold symmetry of curves of class A is that every term in the equation shall be of even degree and that

$$a_{st}=(-1)^t a_{ts}.$$

(This result is not explicitly stated there, but is easily deduced as a corollary from the argument.) This enables us in the present case to write (15) in different form. We replace ϕ by its value 45°.

$$(16)\qquad \sum_{t,\, s=0}^{t,\, s=v}[a_{ts}(\cos^t \theta_1 \sin^s \theta_1+(-1)^t \cos^s \theta_1 \sin^t \theta_1)]$$

$$=-\sum_{t,\, s=0}^{t,\, s=v}\{a_{ts}[\cos^t(\theta_1+45°)\sin^s(\theta_1+45°)+(-1)^t\cos^s(\theta_1+45°)\sin^t(\theta_1+45°)]\},$$

where $t+s=\nu$, ν being an even number and $t \overline{>} s$. It follows that the existence of equation (16) is the necessary and sufficient condition for 8-fold symmetrical curves of class B. By its aid one may determine the equations of 8-fold symmetrical loci.

4. A simplification in constructing the equations in general. If

$$r=p_1^{a_1}\, p_2^{a_2} \ldots p_k^{a_k},$$

where $p_1, p_2, \ldots, p_k$ are different primes, we may evidently proceed as follows to construct the equations of nth degree loci possessing r-fold symmetry:

Construct the equations of nth degree loci possessing symmetry of class A and of orders $p_1^{a_1}, p_2^{a_2}, \ldots, p_k^{a_k}$, respectively. From these construct the most general equation in which the coefficients obey all the limitations imposed in the several equations separately. The result is the most general form of the equation of class A. Proceed similarly for class B.

5. An example. As an illustrative example consider a special case of the seventh degree curve of class B of 10-fold symmetry in the table above. Let $c_1=c_4=0$, $c_2\neq 0$, $c_3\neq 0$. Then the equation is of the form:

$$(17)\ a_1(5x^4y-10x^2y^3+y^5)+a_2(x^2+y^2)(x^5-10x^3y^2+5xy^4)=0,\ a_1\neq 0,\ a_2\neq 0.$$

Transforming to polar coordinates by the substitution $x=\rho\cos\theta$, $y=\rho\sin\theta$, and substituting $\cos 5\theta$ and $\sin 5\theta$ for $\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$ and $5\cos^4\theta\sin\theta-10\cos^2\theta\sin^3\theta+\sin^5\theta$, respectively, the equation becomes

$$a_1\rho^5 \sin 5\,\theta + a_2\rho^7 \cos 5\,\theta = 0.$$

Evidently this may be replaced by the two equations

$$\rho^5 = 0,$$

(18) $$\rho^2 = a \tan 5\,\theta,$$

where $$a = -a_1/a_2.$$

To the former of these two equations corresponds only the origin. But this point is on the locus of the other equation. Hence, so far as plotting the curve is concerned, equation (17) may be replaced by equation (18). Evidently, it consists of five branches, alike except for position. Each branch passes through the origin and has in itself two-fold symmetry with respect to the origin. Moreover, the origin is obviously a point of inflection for each branch, and there are thus five (but only five) points of inflection at the origin. Now, since the curve possesses 10-fold symmetry, singularities not at the origin can enter only by tens. Hence the number of points of inflection is an odd multiple of 5. It is easy to see that there is no cusp at the origin. Hence cusps enter only in tens, if at all.

For the further discussion of singularities we require the following Plücker equations, which are written in the ordinary notation:

(19) $$m = n(n-1) - (2\,\delta + 3\,\rho),$$

(20) $$n = m(m-1) - (2\,\tau + 3\,\iota),$$

(21) $$\iota = 3n(n-2) - (6\,\delta + 8\,\rho),$$

(22) $$\rho = 3m(m-2) - (6\,\tau + 8\,\iota).$$

We now have $n=7$, ι=odd multiple of 5, ρ=multiple of 10, or zero. Then from (21) it may be seen that $6\,\delta + 8\,\rho$ must be an even multiple of 5; that is, a multiple of 10. But ρ is a multiple of 10, or zero; hence δ is a multiple of 5. It is obvious from (20) that $m \overline{\overline{>}}\, 4$; hence from (19) it follows that either δ or ρ is zero; and therefore $\rho = 0$, since the curve under consideration has double points at the origin. Now the locus is of the seventh degree and cannot have as many as ten coincident points; hence, since δ is a multiple of 5, the number of double points at the origin is 5. Therefore, from (19) it follows that $\delta = 5$ or 15, since singularities not at the origin enter only by tens. We shall now determine which of these is the true value.

Suppose that there is a double point not at the origin; and let it be at a distance d from the origin. Then there must be ten such double points at a distance d from the origin. Pass through them a circle with radius d

and center at the origin. Since each double point counts as two points the circle cuts the septic curve in 20 points. But this is impossible, hence there is no double point except at the origin. Therefore $\delta=5$. Then from Plücker's equations: $m=32$, $\iota=75$, $\tau=380$. Hence the curve

$$5x^4y-10x^2y^3+y^5+c(x^2+y^2)(x^5-10x^3y+5xy^4)=0,\ c\neq0.$$

is of class 32, is non-cuspidal, and has five double points at the origin, 75 points of inflection of which five are at the origin, and 380 double tangents. It is obvious that not all the singularities are real.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

330. Proposed by R. D. CARMICHAEL, Princeton, N. J.

An important function in the Theory of Numbers is one defined thus: $f(x)=1$ when $x>0$, $f(x)=0$ when $x=0$, $f(x)=-1$ when $x<0$. Two analytic expressions for $f(x)$ are the following:

$$f(x)=\lim_{n\doteq\infty}x^{1/(2n-1)},\ n=1,\ 2,\ \ldots;\quad f(x)=\lim_{n\doteq\infty}\frac{(x+1)^n-(x+1)^{-n}}{(x+1)^n+(x+1)^{-n}},\ x>-1.$$

It is required to find other non-trigonometric analytic expressions for this function. (There are several representations of $f(x)$ by means of trigonometric functions.)

No solution of this problem has been received.

331. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Extract the square root of $21+6\sqrt{2}+2\sqrt{21}-6\sqrt{3}-6\sqrt{7}-2\sqrt{6}-2\sqrt{14}$ and also of $4\sqrt{2}+2\sqrt{6}-9-4\sqrt{3}$.

Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y., and J. SCHEFFER, A.M., Hagerstown, Md.

(a) Assume the root to be of the form

$$a\sqrt{2}+b\sqrt{3}+c\sqrt{7}+d.$$

Squaring and comparing coefficients, we have